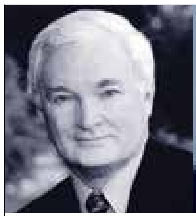
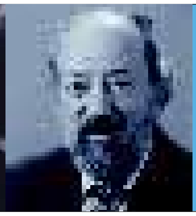




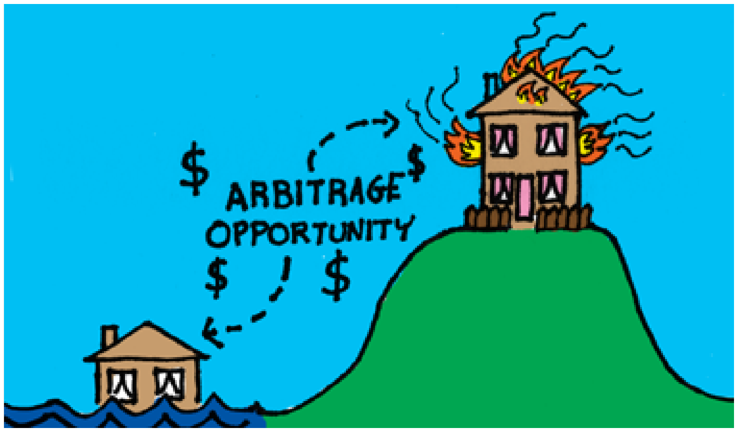
Leonard MacLean and Bill Ziemba

Primer on Arbitrage and Asset Pricing

Back to first principles…

Financial decision making is typically concerned with the amount of investment capital to allocate to various assets or asset classes in various financial markets. The opportunity set can be very complex, with sets of equities, bonds, commodities, derivatives, futures, and currencies changing stochastically and dynamically over time. To consider decisions in a complex market, it is necessary to impose structure. In the abstract, the assets and the participants buying and selling them are parts of a system with underlying economic states. The system's dynamics and the factors defining the states within it have been studied extensively in finance and economics. The dynamics of the market and the behavior of participants determine the trading prices of the various assets in the opportunity set.[1] A simplifying assumption is that the financial market is perfectly competitive. There are conditions which must be present for a perfectly competitive market structure to exist. There must be many participants in the market, none of which is large enough to affect prices. Individuals should be able to buy and sell without restriction. All participants in the market have complete information about prices. In the competitive market, investors are price takers. These assumptions are strong, and in actual financial markets they are not exactly satisfied. However, with the assumed structure, an idealized market can be characterized and that provides a standard by which existing practice can be measured.

If investors are price takers, then a fundamental component of financial decision making is asset pricing. A common approach to asset pricing is to derive equilibrium prices for assets in a competitive market. This can be achieved with a model mapping the abstract states defined by a probability space into prices of assets such as equities and bonds. The capital asset pricing model (CAPM), developed independently by Sharpe (1964), Lintner (1965), Mossin (1966), and Treynor (1961,1962), is a standard for pricing risky assets. Some clarification is provided in Fama (1968). The model proposes that the expected excess return of a risky asset over a riskless asset is proportional to the expected excess return of the market over the riskless asset. The returns on assets are assumed to be normally distributed. In this setting, the financial market is in competitive equilibrium. Consistent with this structure, the optimal investment decisions are determined from the Markowitz (1952, 1959) mean-variance approach. The CAPM is the theoretical basis for much of the sizable index fund business. Dimension Fund Advisors alone manages $250 billion, most of which is passive by following indices like the S&P500 large cap and the Russell2000 small cap.



The CAPM model has a single explanatory variable for asset pricing, the return on the market portfolio, in a simple linear regression. This model has been extended to include other market variables in a multivariate linear regression. For example, following Rosenberg (1974), and Rosenberg, Reid, and Lanstein (1985), Fama and French (1992) have added two explanatory variables: (i) small minus large capitalization; and (ii) high minus low book-to-market ratio.

The equilibrium pricing in the CAPM type models implies that no arbitrage opportunities exist. An arbitrage is a transaction that involves no negative cash flow at any probabilistic or temporal state and a positive cash flow in at least one state; in simple terms, it is the possibility of a risk-free profit at zero cost. The arbitrage pricing theory (APT) for asset pricing, following from an assumption of no arbitrage, was developed by Ross (1976). This theory defines the expected returns on assets with a linear factor model. The theory linking arbitrage to the factor model is presented by Ross (1976). The Ross argument considers a well-diversified portfolio of risky assets which uses no wealth ('free lunch'). The portfolio is essentially independent of noise. If the portfolio has no risk, then the random return is certain, and to avoid disequilibrium the certain return must be zero. The no-arbitrage condition implies that the returns on the assets are defined by a linear relation to a set of common random factors with zero expectation. This type of equilibrium arbitrage argument follows the famous Modigliani and Miller paper (1958), which used arbitrage to argue that a firm's capital structure is irrelevant to the firm's value. It is also part of the reasoning in the Black–Scholes (1973) option pricing model, where a riskless arbitrage position is set equal to the riskless rate of interest.

There are a number of differences between the CAPM and APT theories. The most significant distinction is the 'factors.' In CAPM, the factors/independent variables are manifest market variables (e.g., market index). With APT, the factors are intrinsic (not manifest) variables, whose existence follows from diversification and no arbitrage. It is not required that the APT factors have clear definitions as entities. The APT factors are structural, without implied causation. That is, CAPM: factors → returns; APT: factors ↔ returns. So, the factor model in APT is really a distributional condition on prices following from no arbitrage. The essence of arbitrage is captured in Ross' theory.

There are no assumptions in the APT about the distribution of noise, whereas CAPM assumes normality. However, the use of the factor model in empirical work on pricing does use algorithms, which sometimes assume normality of the factors and returns. The statistical estimation would also suggest definitions/entities for the intrinsic factors, which could further link the CAPM and APT models. Factor models have been used in practice by many analysts (see Jacobs and Levy (1988), Ziemba and Schwartz (1991), and Schwartz and Ziemba (2000)). Companies such as BARRA lease such models.

The APT does not assume the existence of a competitive equilibrium. Disequilibrium can exist in the theory, but it is assumed that in aggregate the returns are uniformly bounded.

The no-arbitrage assumption is a natural condition to expect of a stable financial market. The existence of arbitrage-free prices for assets is linked to the probability measure on which the stochastic process of prices is defined. The fundamental theorem of asset pricing states that:

If $S = \{S_s t \geq 0\}$ are asset prices in a complete financial market, then the following statements are equivalent:

- S does not allow for arbitrage;
- There exists a probability measure which is equivalent to the original underlying measure and the price process is a martingale under the new measure.

A martingale is a stochastic process, where the conditional expected value for the next period equals the current observed value, and does not depend on the history of the process. So, a martingale is a model for a fair process and it is not surprising that the fairness of no arbitrage can be characterized by a martingale measure. Indeed, the Ross (1976) argument establishes the link between arbitrage and a martingale measure using the famous Hahn–Banach theorem. This theorem guarantees the existence of a hyperplane which separates convex sets. In APT, the sets are linear functions of the asset prices: the set such that the claim/value of the combination is less than or equal to zero, and the set of feasible trading strategies. The sets are disjoint and hence feasible investment strategies are arbitrage free. The separating hyperplane generates a risk-neutral probability measure. The assumptions used by Ross on the underlying measure were somewhat limiting. In the case of an infinite probability space, the Ross result only applies to the sup-norm topology. For finite dimensional space, it is not clear that the martingale measure is actually equivalent to the original measure.

These limitations were considered by Harrison and Kreps (1979) and Harrison and Pliska (1981). They extended the fundamental theorem of asset pricing in several ways:

- If the price process is defined on a finite, filtered, probability space, then the market contains no arbitrage possibilities if, and only if, there is an equivalent martingale measure;
- If the price process is defined on a continuous probability space and the market admits 'no free lunch,' then there exists an equivalent martingale measure;
- If the price process is defined on a countably generated probability space, taking values in L^p space, then the 'no free lunch' condition is satisfied if, and only if, there is an equivalent martingale measure satisfying a moment condition.

Although the work of Kreps and colleagues made significant contributions to the theory of arbitrage pricing, there were still assumptions which limited the applicability. Ideally, a more economically natural condition could replace the moment condition on the martingale measure. Delbaen and Schachermayer (2006) discuss many open questions. One particular advance links the existence of an equivalent martingale measure in processes in continuous time or infinite discrete time to a condition of 'no free lunch with bounded risk.' Unfortunately, this result does not hold for price processes which are semi-martingales. Furthermore, there are strong mathematical and economic reasons to assume that the price process is a semi-martingale. In that setting, the no free lunch with bounded risk is replaced by a 'no free lunch with vanishing risk,' where risk disappears in the limit. The latter is stronger than the former, but is weaker than a no-arbitrage condition. Schachermayer (2010a) and Delbaen and Schachermayer (2006) have a general statement of the fundamental theorem:

"*Assume the price process is a locally bounded real-valued semi-martingale. There is a martingale measure which is equivalent to the original measure if and only if the price process satisfies the no free lunch with vanishing risk condition.*"

Yan (1998) brought the results even closer to the desired form. The concept of allowable trading strategies was introduced, where the trader remains liquid during the trading interval. The Yan formulation yields the result:

"*Let the price process be a positive semi-martingale. There is a martingale measure which is equivalent to the original measure if and only if the price process satisfies the no free lunch with vanishing risk condition with respect to allowable trading strategies.*"

Another term for an equivalent martingale measure is a risk-neutral measure. Prices of assets depend on their risk, with a premium required for riskier assets. The advantage of the equivalent martingale or risk-neutral measure is that risk premia are incorporated into the expectation with respect to that measure. Under the risk-neutral measure, all assets have the same expected value – the risk-free rate. The stock price process discounted by the risk-free rate is a martingale under the risk-neutral measure. This simplification is important in the valuation of assets such as options and is a component of the famous Black–Scholes (1973) formula. Of course, the risk-neutral measure is an artificial concept, with important implications for the theory of pricing (Schachermayer, 2010b). The actual risk-neutral measure used for price adjustment must be determined from economic reasoning.

The separating hyperplane arguments underlying the results linking arbitrage and no free lunch to martingale measures have an analogy in theorems of the alternative for discrete time and discrete space arbitrage pricing models. In theorems of the alternative, competing systems of equalities/inequalities are posed, with only one of the two systems having a solution. A famous such theorem is due to Tucker (1956). Kallio and Ziemba (2007) used Tucker's theorem of the alternative to derive known and some new arbitrage pricing results. The competing systems define arbitrage on the one hand and the existence of risk-neutral probabilities on the other. For a frictionless market, the fundamental theorem of asset pricing is established using matrix arguments for the discrete time and discrete space price process:

"*If at each stage an asset exists with strictly positive return (there exists a trading strategy), then arbitrage does not exist if and only if there exists an equivalent martingale measure.*"

Although the discrete time and space setting is limiting, it is used in practice as an approximation to the continuous process. Obviously, there are considerable computational advantages with a discrete process, and assumptions required for its implementation are few. In the general setting, the fundamental theorem posits the existence of a risk-neutral measure. Actually, finding such a measure requires additional assumptions. In the discrete setting, the equations for calculating the probabilities in the measure can be solved. This is analogous to the option pricing models, where in the Black–Scholes approach strong distribution assumptions are required to get the pricing formula, but the binomial lattice approach obtains option prices with a linear programming algorithm. Even from a theory perspective, the discrete time and space extension to more complex financial markets is feasible, as the mathematics is based on systems of equations. In Kallio and Ziemba (2007), the equivalence between no arbitrage and the existence of a martingale measure is extended to markets with various imperfections.

The no-arbitrage condition is fundamental to much of the theory of efficient capital markets. However, it is important to acknowledge the existence of arbitrage opportunities in actual markets. Examples are the Nikkei put warrant arbitrage discussed by Shaw et al. (1995), and the race track arbitrages discussed by Hausch and Ziemba (1990a,b). Investors exhibit behavioral biases which can lead to mispricing and arbitrage. Usually, over/under pricing is temporary, but correctly identifying those events and using them for financial advantage has attracted attention.

An illustration of the behavioral bias leading to arbitrage is given in the paper by MacLean et al. (2013). They consider the bias in investment decisions in foreign currencies from the perspective of the state of the financial market. It is hypothesized that the market is classified into a number of regimes, defined by characteristics such as yield differentials, credit spreads, financial ratios, and volatility. Depending on market characteristics, investors present biases in decisions by overreacting to information. The effect is that actual decision behavior deviates from rational expectations. However, if the bias is a natural reaction to information, then conditioning on the prevailing regime will account for the bias and result in accurate predictions of decision behavior. The efficient market condition for currencies is interest rate parity, where parity implies that the mean excess hedged returns are zero and currencies do not have a risk premium. The parity condition does not hold for actual currency returns. Assuming a persistent behavioral decision bias in a regime, the mean excess hedged returns are equal but not necessarily zero within each regime, and are different across regimes.

This regime-dependent behavior effect is tested with currency data from five major currencies covering the period 2002–2007, inclusive. It is found that the data exhibit the anticipated regime structure. That is, investment decisions have a persistent bias depending on the market structure. Using a regime-dependent equilibrium portfolio, it is observed that portfolio returns are a close match to market indices. The market indices have the decision bias imbedded, whereas the equilibrium portfolio explicitly models the decision bias and the matching is a confirmation of the model.

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ENDNOTE

1. This column is modified from Introduction A in Part I of our handbook, The Fundamentals of Financial Decision Making, World Scientific, 2013, where readers can find many of the papers cited here plus other papers and discussion on financial decision making..

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