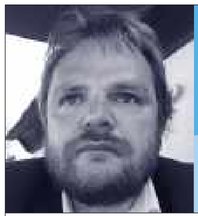


Rolf Poulsen

Binary Backwards

A Twitter exchange leads to a useful discussion on option modeling. So, social media isn't all bad…

Anybody who writes exams or performs job interviews knows the value of questions. If they are based on true stories or statements, even better. To my delight this showed up in my Twitter timeline (Figure 1).

Figure 1: A delightful Twitter surprise

(Let us assume @FMTrader1 describes an at-the-money down-binary (or digital) option with one week (five business days; 5/252 years) to expiry.)

Starter for ten, Q1: *What is the initial price of the digital option?*

The payoff is either 1 or 0, thus 1 is the only case with a positive rate of return, so the price, p, must solve $(1-p)/p = 0.7$, i.e., $p = 0.588$.

Going into modeling, Q2: *Is that price consistent with the Black–Scholes model?* In the Black–Scholes model, the price of this at-the-money down-binary option is

$$1-\Phi\left(\frac{\tau\left(r-d-\sigma^2/2\right)}{\sigma\sqrt{\tau}}\right),$$

which goes rapidly to ½ for $\tau \to 0$, meaning that with one week to expiry we'd need extreme parameter assumptions to generate a price of 0.588. So, in a word: No. (The question can also be phrased such that it works for students who've only heard of the standard binomial model, but either the question or the answer becomes much less elegant.)

Feeling smug, I sent out the questions to people in the quantitative finance community.

One of the recipients, let's call him KwantDaddy, chipped in with Q3 (at 10:39): *Is it consistent with a jump diffusion model (à la the Merton model)? If the candidate can answer this correctly, we will make an offer.*

Rolf (at 11:16 pm): With a Poisson-jump-component, the distribution of the change in (the log of) the stock price can be made asymmetric also at short time-steps. The option price is the probability of the change being negative. So, a negative average jump size should do the trick here.

KwantDaddy (at 11:21 pm): Is that your final answer?

Rolf (at 12:39 am; verbatim from my email; sic): Aaaah, the prob' of the goes to 0 like dt, and as option is binary, we get price effects the jump size not scaling by the length of the time-step. I now think, no, the price still -> 0.5 as dt-> 0.

That answer seemed to satisfy KwantDaddy. And it is indeed correct as can be seen by direct inspection of the call-price formula in the Merton jump-diffusion model. An analysis (in which Uwe Wystup and Antoine Savine partook in various ways) along the following lines then ensued.

Let's say we construct a strike spread: Buy $\frac{1}{\varepsilon}$ calls with strike $K-\varepsilon$, sell $\frac{1}{\varepsilon}$ calls with strike K. As $\varepsilon \to 0$, this approximates up-digital call payoff; see Figure 2.

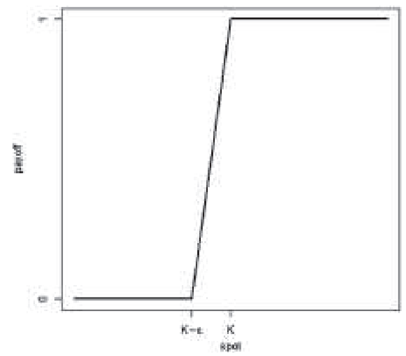
The cost of the strike spread is $\frac{\mathrm{Call}(K)-\mathrm{Call}(K,\varepsilon)}{\varepsilon}$, so by definition of a derivative we have

$$\text{Up-Digital} = \frac{\partial\,\mathrm{Call}}{\partial K}$$

Any arbitrage-free call price – irrespective of which market or model may have generated it – can be expressed via its implied Black–Scholes volatility, $\sigma^{imp}(K)$, where we make the strike-dependence explicit in the notation but suppress everything else. Hence, $\mathrm{Call}(K) = \mathrm{Call}^{B-S}(K, \sigma^{imp}(K))$, and by the chain rule for differentiation

$$\text{Up-Digital} = \Phi(d_2) + \mathrm{Vega}^{B-S} * \frac{\partial \sigma^{imp}}{\partial K}, \qquad (*)$$

Figure 2: Strike spread approximating up-digital payoff



where, as usual, $d_{1,2} = \frac{\ln\left(\frac{S}{K}\right)+\left((r-d)+,-\frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}$, $\text{Vega}^{B-S} = Se^{-\frac{1}{2}d_1^2}\sqrt{\tau}$ and the last term represents the slope of the implied volatility surface.[1]

Three things are noteworthy about Equation (*):

1) We have made no modeling assumptions (Black–Scholes or otherwise), nor any assumptions about strike (like at-the-money) or expiry (like short).
2) For short expiries, the at-the-money Black–Scholes Vega behaves like $\sqrt{\tau}$. This fairly slow convergence rate means that at small, but non-infinitesimal expiries there is more flexibility to create digital options prices different from ½ via a skew in implied volatilities.[2] For example, if rates are 0, spot = strike = 1 and options with strikes (0.9, 1, 1.1) trade at implied volatilities (0.26, 0.2, 0.14), then a one-week, at-the-money down-digital would have a price of 0.59.
3) For the short expiry limit not to be ½, implied volatility must be quite wild; it needs not only to diverge, but also to do so in a way bad enough to offset the $\sqrt{\tau}$-factor. However, applying this result is not quite as simple as it first looks. Models (with SABR as an exception) are (for good reason) not specified directly in terms of their implied volatilities; rather, we have to first find the model's options prices and study the behavior of the nonlinear, non-explicit transformation that is implied volatility (in fact, its strike derivative). For any diffusion-based model as well any finite intensity jump model, a finite of limit can be shown to exist. However, it can also be shown that arbitrage-free models do exist in which the $\sqrt{\tau}\to 0$-limit of (*) is not 0 (see Roper & Rutkowski (2009) or Jacquier & She (2016)). Recently, rough volatility models have gained traction in quantitative finance circles; see, for instance, Gatheral, Jaisson & Rosenbaum (2014). In these models, the short term skew behaves as $\tau^{H-1/2}$, where the so-called Hurst exponent is typically estimated at 0.1–0.2, meaning that while the skew explodes, the limit of (*) is still ½.

About the Author

Rolf Poulsen is a professor of Mathematical Finance in the Department of Mathematical Sciences at the University of Copenhagen. His main research interest is quantitative methods for pricing and hedging of derivatives. He will talk about exchange rate markets at length to all who will listen – and some who won't.

ENDNOTE

1. Or, to be precise: the Black–Scholes implied volatility surface. Some sources, such as Andreasen and Huge (2012), prefer to work with Bachelier implied volatility. I guess this is for two reasons: partly because the Bachelier model is born with the negative skew that is predominant in interest rate options markets, and partly because constant coefficient differential equations are simply simpler to work with.
2. At first thought, one might think that the short-expiry skew flattens out in diffusions model. But because the calculation of implied volatility serves as a magnifying glass for differences to the Black-Scholes model, this does not happen. However, careless discretization in numerical calculations may lead to spurious flattening. An example based on a true story: Simulation of the CEV-model with the log-Euler scheme and step-size dt = 1/12 gives a perfectly reasonable skew for 1-year options, but a completely flat skew for 1-month options.

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