A Markovian Model of Default Interactions: Comments and Extensions

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1 Model Analysis

This article analyses Davis and Lo (2001b) enhanced risk model, which is a dynamic version of the popular market model of infectious defaults of Davis and Lo (2001a). For all details regarding the enhanced risk model we refer the reader to the original article of Davis and Lo (2001b). In this article we review the main conclusions of the model and obtain a closedform solution that should be valuable in practice.

As shown in the original article, the model reduces to a set of simultaneous ordinary differential equations (ODEs). Solving this system of equations numerically is a challenging problem for large diversified portfolios. We propose an alternative method of dealing with the problem. In addition to this, we analyse the behaviour of the portfolio in the limit of a large relaxation time and establish useful results that give thorough understanding of the underlying dynamics.

The enhanced risk model of Davis and Lo (2001b) is a two-state Markovian model, where a bond portfolio migrates back and forth between normal and enhanced risk states over time. Initially the bond portfolio is in the normal risk state, where all bonds have the same default intensity $\lambda \ge 0$. Following a default of a single obligor, the portfolio migrates to the enhanced risk state and all the remaining bonds

become more risky. In this state the default intensity rate is increased to $a\lambda$, where a > 1. After an exponentially-distributed period of time characterised by the parameter $\mu \ge 0$, the portfolio returns to the normal risk state. One may think of the enhanced state as a period of crisis or market turbulence, when contagious defaults become problematic in a portfolio context.

An attractive feature of the enhanced risk model is that the migration of the portfolio between the two states is described endogeneously through the default event trigger by a single obligor, rather than being related to some vaguely defined set of external market factors.

As shown by Davis and Lo (2001b) the portfolio follows a Markovian process on the state space

$$E = \{(i, k) : i \in \{0, 1\}, k \in \{0, 1, \dots, N\}\}$$

where (i = 0,1) denotes the normal and enhanced risk states respectively and k is a number of undefaulted securities.

The initial position for the portfolio is (0, N). Let $p_k^i(t)$ denote the probability of being in state (i, k) at time t. It is easy to show that the portfolio dynamics (see Figure 1) is described by the forward matrix equation for the probability density, which is equivalent to the following set of ODEs,

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$$\frac{dp_k^1}{dt} = p_{k+1}^0 \lambda(k+1) + p_{k+1}^1 a\lambda(k+1) - p_k^1(\mu + a\lambda k)$$

$$\frac{dp_k^0}{dt} = -p_k^0 \lambda k + p_k^1 \mu$$
(1)

In order to find the forward density function, one needs to solve the set of 2N equations with the initial condition $p_N^0(0) = 1$, $p_k^0(0) = 0$, k < Nand $p_k^1 = 0$, $k \le N$. The authors suggested employing a numerical integration procedure (e.g., Runge-Kutta method) to tackle this set of equations. We feel that for large or even medium values of N this approach may be computationally intensive. In addition, numerical methods usually provide only a limited understanding of the underlying dynamics. We will exploit an alternative methodology to finding a solution for this problem, which is outlined in Section 2. In Section 3 we develop approximation methods valid in the limit of fast relaxation times (when the total time spent in the enhanced state is short relative to the time spent in the normal state). In Section 4 we consider numerical results.

2 Numerical Approach

Specifically, we will use the Master equation approach for the survival probability P(t, i, k), where *i* represents the state and *k* is the number of undefaulted bonds (out of the initial *N*) at time *t*. The survival probability satisfies the following system of Master equations

$$P(t + \Delta t, 0, k) = P(t, 0, k)(1 - \lambda \Delta t)^{k} + P(t, 1, k)\mu \Delta t$$

$$P(t + \Delta t, 1, k) = P(t, 0, k + 1)C_{1}^{k+1}\lambda \Delta t(1 - \lambda \Delta t)^{k}$$

$$+ P(t, 1, k + 1)C_{1}^{k+1}a\lambda \Delta t(1 - a\lambda \Delta t)^{k}$$

$$+ P(t, 1, k)(1 - a\lambda \Delta t)^{k}(1 - \mu \Delta t)$$

where Δt is a small time step.

The differential form of the Master equation can be obtained by expanding the left hand side of the system and retaining terms of the first order of magnitude in Δt ; this yields

$$\begin{aligned} \frac{\partial P}{\partial t}(t, 0, k) &= -P(t, 0, k)\lambda k + P(t, 1, k)\mu \\ \frac{\partial P}{\partial t}(t, 1, k) &= P(t, 0, k+1)\lambda(k+1) + P(t, 1, k+1)a\lambda(k+1) \\ &- P(t, 1, k)(a\lambda k + \mu) \\ P(0, 0, k) &= \delta_{kN}, \quad P(0, 1, k) = 0, \ \forall k \end{aligned}$$

where δ_{kN} is the Kronecker symbol reflecting that initially we had exactly N bonds in our portfolio. To resolve this system of equations we use the method of moment generating functions. We introduce

$$\phi^{-}(s,t) = \sum_{k=0}^{N} s^{k} P(t,0,k), \quad 0 \le s \le 1$$

$$\phi^{+}(s,t) = \sum_{k=0}^{N} s^{k} P(t,1,k), \quad 0 \le s \le 1$$

$$\phi(s,t) = \sum_{k=0}^{N} s^{k} P(t,k), \text{ where } P(t,k) = P(t,0,k) + P(t,1,k)$$

$$\phi(s,t) = \phi^{-}(s,t) + \phi^{+}(s,t)$$

In terms of the moment generating functions we can write the Master equations as follows

$$\frac{\partial \phi}{\partial t}^{-} = -\lambda s \frac{\partial \phi}{\partial s}^{-} + \mu \phi^{+} \tag{2}$$

$$\frac{\partial \phi}{\partial t}^{+} = -\lambda a(s-1)\frac{\partial \phi}{\partial s}^{+} + \lambda \frac{\partial \phi}{\partial s}^{-} - \mu \phi^{+}$$
(3)
$$\phi^{+}(s,0) = 0, \quad \phi^{-}(s,0) = s^{N}$$

This hyperbolic system has two distinct families of characteristics

characteristic 1: $s_1(t) = s_T e^{-\lambda(T-t)}, t \in [0, T]$ characteristic 2: $s_2(t) = 1 - (1 - s_T)e^{-a\lambda(T-t)}, t \in [0, T]$

which connect any point (s_T, T) $(0 \le s_T \le 1)$ in the plane with initial conditions. Along these characteristics the hyperbolic system transforms into the system of ODEs

$$e^{-\mu t} \frac{d}{dt} \{ e^{\mu t} \phi^+(s_1(t), t) \} = \frac{d}{dt} \{ \phi(s_1(t), t) \}$$
$$\frac{d}{dt} \{ \phi(s_2(t), t) \} = -(a-1)(1-s_2(t))e^{-\mu t} \frac{d}{dt} \{ e^{\mu t} \phi^+(s_2(t), t) \}$$

This can be easily integrated numerically. Indeed, if the system had constant (or slowly varying) coefficients, $K_1 = e^{-\mu t}$ and $K_2 = -(a - 1)(1 - s_2(t))e^{-\mu t}$, then integration along the characteristics would yield

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$$K_{1}\{e^{\mu T}\phi^{+}(s_{T},T) - \phi^{+}(s_{1}(0),0)\} = \{\phi(s_{T},T) - \phi(s_{1}(0),0)\}$$

$$\{\phi(s_{T},T) - \phi(s_{2}(0),0)\} = K_{2}\{e^{\mu T}\phi^{+}(s_{T},T) - \phi^{+}(s_{2}(0),0)\}$$

where $s_{1}(0) = s_{T}e^{-\lambda T}$, $s_{2}(0) = 1 - (1 - s_{T})e^{-a\lambda T}$ (4)

Since $\phi^+(s, 0) = 0$ and $\phi(s, 0) = s^N$, we have a simple system of linear equations for the unknowns $\phi^+(s_T, T)$ and $\phi(s_T, T)$, which has a non-degenerate determinant.

In reality, however, (like in our case) $K_1 = K_1(t)$ and $K_2 = K_2(t)$ can not always be assumed slowly varying functions on [0, T]. Usually in such a case we can consider a partition of the time segment $0 < t_1 < t_2 < t_3 < \ldots < T$, assuming that for each subsegment $[t_k, t_{k+1}]$ the coefficients are slowly varying and can be proxied by their mid-values. Hence, results (4) can be applied for each subsegment. Using the backward induction principle we connect the point (s_T, T) with the initial conditions as shown in Figure 2. In our problem (for times of the order of $O(1/\lambda)$) we may just need several intermediate steps to obtain a desired accuracy. The methodology for solving hyperbolic systems is well known, see for instance Ockendon et al. (1999, p.49). Once we have obtained $\phi(s_T, T)$ for various values of s_T (ideally we need N data points) we can fit a standard Lagrange polynomial. If the number of points is less than N, we can use a coarser grid for our probability density function dividing the space of outcomes into buckets, rather than registering each individual default. In any case the coefficients of this interpolation polynomial are the survival probabilities we are looking for. Hence, we have replaced a numerical integration approach proposed in the original article by a polynomial fitting procedure. Remember that numerical integration of (1) may require very fine time step as N gets large, due to the increase in the magnitude of the coefficients in this set of simultaneous ODEs.

Now we outline how a coarser grid can be constructed. Consider an example where we have a well diversified portfolio consisting of N = 200 exchangeable bonds. Building an interpolation polynomial of the degree N is going to be hard, whichever approach is used. Let us assume that we are trying to evaluate the likelihood of the first loss tranche being completely wiped out. The thickness of the first loss tranche is assumed to be 5%. This



Figure 2: Solving the system by means of characteristics (a two-step approach is illustrated)

tranche will be wiped out if 10 bonds default over a given period of time assuming zero recovery value. Hence, it is not necessary to consider each default individually, instead we can consider them in increments of 10. Therefore, we need to build a polynomial of the degree 20 only, which is not a difficult problem. Let $\{s_T(i)\}_{i=1}^{20}$ be our set of 20 points, where we have evaluated the original moment generating function $\phi_N(s_T(i), T)$ that corresponds to the *N*-point distribution. The reduced moment generating function that correspond to the 20-bucket distribution $\phi_{20}(s, T)$ would satisfy the following conditions (based on the scaling principle)

$$\phi_{20}(s_T(i)^{10}, T) = \phi_N(s_T(i), T)$$
 for $i = \overline{1, 20}$

Constructing the interpolation polynomial $\phi_{20}(s, T)$ based on 20 points is trivial.

3 Approximation Methods

As mentioned before, our bond portfolio follows a Markovian process on the extended space *E*. In other words, at each moment in time we have to register both the state of the portfolio (normal or enhanced) and the number of bonds remaining in the portfolio. When we price various contingent claims (e.g., collaterised debt obligations, etc.), the knowledge of the portfolio state at maturity of the product is irrelevant in most circumstances, as the payoff function has no explicit state dependence.

Intuitively, we would favour such values of the parameters (λ, μ) that the total time spent in the enhanced state is relatively short when compared to the time spent in the normal state. We reckon that the enhanced state should be associated with periods of market turbulence, when correlated defaults become particularly problematic, rather than being another regular state. We can anticipate that this would occur when the relaxation time between the states μ^{-1} is much smaller than the default time λ^{-1} . In other words, we require that

$$\frac{\mu}{\lambda} \gg 1$$
 (5)

Let $\hat{\phi}^+$ and $\hat{\phi}^-$ be typical values for ϕ^+ and ϕ^- , to be exact, say $\hat{\phi}^+ = \phi^+(1, \lambda^{-1})$ and $\hat{\phi}^- = \phi^-(1, \lambda^{-1})$. We assume that the balance between the ratio of the relaxation parameters and times spent in both states takes place, so that

$$\frac{\mu}{\lambda} \cdot \frac{\dot{\phi}^+}{\dot{\phi}^-} = O(1) \tag{6}$$

where O(1) denotes an order one variable. Indeed, as μ increases (a faster relaxation time) we are going to spent less time in the enhanced state. As λ decreases (less defaults) we are going to spent less time in the enhanced state again. We will construct a solution that satisfies this balance criteria by means of an asymptotic expansion.

As we discussed earlier, the portfolio process is Markovian on the extended space *E*, but it will not be Markovian if we

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exclude information regarding the state. Asymptotic analyses induced by condition (5) effectively constructs a Markovian approximation to a non-Markovian process valid for fast relaxation times (short term memory effects are neglected). In physics this approach is known as Adiabatic Elimination of Fast Variables, for instance, see Gardiner (1992, p.195) where a similar approach is applied to Langevin's equation.

Further we proceed with original system of equations (2) (3). We introduce the non-dimensional time $\tau = \lambda t$. By adding up (2) and (3) we obtain the equation for ϕ that we ultimately want to solve for

$$\frac{\partial \phi}{\partial \tau} + (s-1)\frac{\partial \phi}{\partial s} = -(a-1)(s-1)\frac{\partial \phi}{\partial s}^+$$
(7)

$$\phi^{+} = \frac{\lambda}{\mu} \left(\frac{\partial \phi^{-}}{\partial \tau} + s \frac{\partial \phi^{-}}{\partial s} \right) \tag{8}$$

 $\phi^- = \phi - \phi^+ \tag{9}$

To be rigorous with our asymptotic expansion, we have to scale ϕ^+ and ϕ^- with their typical values. For the ease of the exposition, however, we will do this implicitly, as the balance is obvious in our case.

Now we pose a formal Poincaré expansions for the moment generating functions

$$\phi(s,t) = \phi_0(s,t) + \frac{\lambda}{\mu}\phi_1(s,t) + \left(\frac{\lambda}{\mu}\right)^2 \phi_2(s,t) + \cdots$$
$$\phi^+(s,t) = \phi_0^+(s,t) + \frac{\lambda}{\mu}\phi_1^+(s,t) + \left(\frac{\lambda}{\mu}\right)^2 \phi_2^+(s,t) + \cdots$$
$$\phi^-(s,t) = \phi_0^-(s,t) + \frac{\lambda}{\mu}\phi_1^-(s,t) + \left(\frac{\lambda}{\mu}\right)^2 \phi_2^-(s,t) + \cdots$$

The balance equation (6) and equation (9) imply that the leading order term $\phi_0^+(s, t) \equiv 0$ and $\phi_0(s, t) \equiv \phi_0^-(s, t)$. Equation (7) yields the leading order moment generating function,

$$\frac{\partial \phi_0}{\partial \tau} + (s-1)\frac{\partial \phi_0}{\partial s} = 0, \text{ with } \phi_0(s,0) = s^N$$
(10)

Hence,

$$\phi_0(s,\tau) = [1 - (1 - s)e^{-\tau}]^N \tag{11}$$

This is the moment generating function for the binomial distribution. In other words, condition (5) ensures that to the leading order of magnitude our portfolio process follows a binomial distribution assuming independence of obligors each of which has a constant default intensity λ .

Next we are going to find out correcting terms to our expansion. From (8) taking into account that $\phi_0(s, t) \equiv \phi_0^-(s, t)$ we obtain

$$\phi_1^+ = \frac{\partial \phi_0}{\partial \tau} + s \frac{\partial \phi_0}{\partial s} = \frac{\partial \phi_0}{\partial s}$$

The latter equality follows from equation (10). Now we substitute for ϕ_1^+ in equation (7) to obtain

$\frac{\partial \phi_1}{\partial \tau} + (s-1)\frac{\partial \phi_1}{\partial s} = -(a-1)(s-1)\frac{\partial^2 \phi_0}{\partial s^2}, \text{ with } \phi_1(s,0) = 0 \quad (12)$

This equation will be easily solved below.

The directional derivative of the leading order moment generating function, $\phi_0 = \phi_0(s, \tau)$, is zero along the characteristics, as follows from (10). Equation (12) implies that the directional derivative of the first order term $\phi_1 = \phi_1(s, \tau)$ is driven by the difference of the intensities of the normal and enhanced states, a - 1, and the gamma of the moment generating function of the binomial distribution. Obviously, if a = 1, we can ignore higher order terms due to the initial condition $\phi_k(s, 0) = 0$ for $k \ge 1$.

We look for the solution of equation (12) in the form of a polynomial of the degree N,

$$\phi_1(s, \tau) = \sum_{k=0}^N a_k(\tau)(s-1)^k$$

By direct substitution we find that

$$a_k(\tau) = (a-1)C_k^N k(N-k) \{e^{-(k+1)\tau} - e^{-k\tau}\}$$

Hence, we have found the first order approximation term. Likewise higher order terms can be obtained from the iterative procedure described below for $m \ge 1$.

We know ϕ_1 and ϕ_1^+ . From the first equation below we calculate ϕ_1^- , then the second equation yields ϕ_2^+ and finally the last equation enables us to obtain ϕ_2 . In principle we can repeat this procedure until the desired accuracy is achieved.

$$\phi_m^- = \phi_m - \phi_m^+$$

$$\phi_{m+1}^+ = \frac{\partial \phi_m^-}{\partial \tau} + s \frac{\partial \phi_m^-}{\partial s}$$

$$\frac{\partial \phi_m}{\partial \tau} + (s-1) \frac{\partial \phi_m}{\partial s} = -(a-1)(s-1) \frac{\partial \phi_m^+}{\partial s}$$

In this article for simplicity of the exposition we will stop at the first order of approximation, as it provides a good understanding of the dynamics of the underlying system. Hence,

$$\phi(s,t) = \sum_{k=0}^{N} C_k^N \left\{ e^{-\lambda kt} + \frac{\lambda}{\mu} (a-1)k(N-k)[e^{-\lambda(k+1)t} - e^{-\lambda kt}] \right\}$$
$$(s-1)^k + O\left(\frac{\lambda}{\mu}\right)^2$$

and the survival probability of having m bonds not having defaulted out of the total N over the time period [0, t] is given by

$$P(N, m, t) \approx C_m^N \sum_{k=0}^{N-m} C_k^{N-m} (-1)^k c_{k+m}(t), \quad \text{where}$$

$$c_j(t) = \left\{ e^{-\lambda jt} + \frac{\lambda}{\mu} (a-1)j(N-j)[e^{-\lambda(j+1)t} - e^{-\lambda jt}] \right\}$$
(13)

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One may notice that $P(N, N, t) = e^{-\lambda N t}$ does not depend on the parameters related to the enhanced risk state, *a* and μ .

4 Results

In Figure 3, we illustrate P(N, m, t) for both the classical Binomial (a = 1) and enhanced risk (13) distributions. We observe the tail widening effect, as we increase the state enhancement parameter *a*. At the same time, however, we have to readjust the intensity parameter λ , in

T = 15-year horizon 25.00% Portfolio of N = 30 bonds Recovery μ = 50% (2/3 is a probability of recovery over 2-year period) 20.00 15.00 Probability $a=2, \lambda = 0.6$ ■ a=1, λ = 0.8 10.00 5.00 0.00 30 29 28 27 26 25 24 23 22 21 20 19 18 17 16 15 14 13 12 11 3 10 6 5 Number of bonds not defaulted





Figure 4: Expected Portfolio Value

order for both distributions to have the same expected value. The widening of the tail is closely related to periods of market turbulence, when correlated defaults can impact most senior tranches of a CDO. If λ is kept constant, while *a* is increased, then both the mean and the variance of the distribution would increase to reflect higher realised defaults over time. We have the following approximation for the expected portfolio value

$$E(\Pi) \approx Ne^{-\lambda t} \left(1 - \frac{\lambda}{\mu}(N-1)(a-1)(1-e^{-\lambda t})\right)$$

In Figure 4 we illustrate $E(\Pi)$ as time elapses. For comparison we also show the expected portfolio value assuming that portfolio stays either in the normal or enhanced states all the time. Obviously in the normal risk-state portfolio's expected value is $N \exp(-\lambda t)$ and in the enchanced risk state (assuming that portfolio stays there all the time) is $N \exp(-a\lambda t)$. As expected, the actual portfolio value lies between these two boundaries, depending upon the magnitude of the relaxation parameter μ .

Further, we consider our asymptotic expansion as *N* gets large. In fact, for large portfolios the probability of the default event (triggered by any obligor from the portfolio) increases linearly with *N*. Hence, for a constant relaxation parameter μ , the portfolio will spend most of the time in the enhanced state. In other words, the model would penalise large portfolios. In order to resolve this issue we would have to scale μ with the portfolio size by introducing $\hat{\mu} = \mu/N$. Therefore, we really have to require that $\hat{\mu}/\lambda \gg 1$.

Similarly from the moment generating function we find portfolio's variance,

$$\operatorname{VaR}(\Pi) \approx N(N-1)e^{-2\lambda t} \left(1 - 2\frac{\lambda}{\hat{\mu}} \frac{N-2}{N}(a-1)(1-e^{-\lambda t})\right) + E(\Pi) - E(\Pi)^2$$

In Figure 5 we observe that portfolio's variance increases, as the enhancement parameter *a* gets larger. Here, as in Figure 3, we readjust λ accordingly, in order to make distributions comparable. The increase in the portfolio variance is mostly a reflection of the tail widening effect observed in Figure 3.

Finally, in Figure 6 we compare the asymptotical solution of Section 3 against the numerical solution of Section 2. We can see that the asymptotical solution developed in this article provides a good approximation





Figure 5: Portfolio Variance



Figure 6: Numerical and Asymptotical Solutions Compared

and hence is a valuable tool for pricing exotic credit structures in a dynamical framework.

5 Conclusions

In this article we have conducted a comprehensive analyses of the enhanced risk model. We have proposed an efficient numerical method for finding the survival probability. In addition to this we have obtained a closedform asymptotical solution valid in the limit of fast relaxation times. This approximation can be very useful in practice to assess the impact of various parameters on the dynamics of the portfolio. We have shown that the relaxation parameter μ should be also related to the size of the portfolio and not only to the speed of the market recovery. This becomes particularly important when large diversified portfolios are considered, as keeping μ fixed, while increasing the size of the portfolio would unduly penalise diversification benefits.

FOOTNOTE & REFERENCES

1. The values of the characteristic function for s = 1 imply the probability of being in that state at a given moment of time.

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