A Conditional Valuation Approach for Path-Dependent Instruments

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1. Introduction

In an effort to improve credit risk management, financial institutions have developed various measures to manage their exposure to counterparty risk. One important measure of counterparty risk is potential future exposure (PFE), which is a percentile (typically 95 or 99 percent) of the distribution of exposures at any particular future date. Credit exposure is the amount a bank can potentially lose in the event that one of its counterparties defaults. The measurement of exposure for derivative products is very important because it is used not only to set up trading limits but also as an essential input to economic and regulatory capital. The internal economic capital models used by most technologically advanced banks require the calculation of the distribution of the exposure at specified future times. For banks intending to use the internal model method in the new Basel II revised framework on trading activities Basel Committee (2005), specific exposure measures such as the expected exposure (EE) and expected positive exposure (EPE)[1] are required in the calculation of the regulatory capital.

This paper focuses on the methodology for calculating the potential future exposure of path-dependent derivative instruments. Unlike loan products, the value of derivatives and other market-driven contracts can change significantly over time as a result of market movements. This may lead to a potential credit exposure with the trading counterparty should it default in the future and its transactions have a positive market value to the bank. Most banks use a variety of methods to manage such risk at the counterparty level, which may include limits on potential exposures, netting, collateral agreements, and early termination agreements. Since most credit limits are based on potential exposure, it is important for a bank to have robust and accurate risk models, as well as systems infrastructure, to quantify the potential exposures of its derivatives positions.

The potential exposure at a future time is defined in this paper as some percentile on the distribution of instrument priced at today for many different scenarios at the future time, and it is not on the distribution of instrument prices realized at the future time. There are two main components in calculating credit exposures on a transaction: scenario generation and instrument valuation. The scenario generation is a simulation process that generates the future scenarios of various market risk factors at different future times (or simulation dates as termed in this paper). Similar to front office transaction pricing, exposure calculation also requires a valuation model in order to calculate the value of a transaction over different times in the future. The similarity stops here, however, as the calculation of credit exposure requires modeling that may not be consistent with the front office valuation model, particularly the scenario generation process. For credit exposure, our concern is on the potential future market value of a transaction. Future market scenarios are usually generated using evolution models of the underlying risk factors under the "real measure" instead of the "risk-neutral" pricing measure used in the calibration of the front office models. Furthermore, the calculation of credit exposure requires a valuation model to value the trade not only at the current time, but also to "Value-at-Future" (VaF)[2] or price the trade consistently across different times in the future (see illustration below).

The calculation of credit exposure relies heavily on simulation[3], especially when counterparty's portfolio is dependent on multiple risk factors. Because of the computational intensity required to calculate counterparty exposures, especially for a bank with a large portfolio, compromises are usually made with regards to the number of simulation times or the number of scenarios. For example, the simulation times (also called "time buckets") used by most banks to calculate credit exposure usually have daily or weekly intervals up to a month, then monthly up to a year and yearly up to five years, etc. We generate market scenarios across these simulation times. The basic problem in valuing path-dependent instruments

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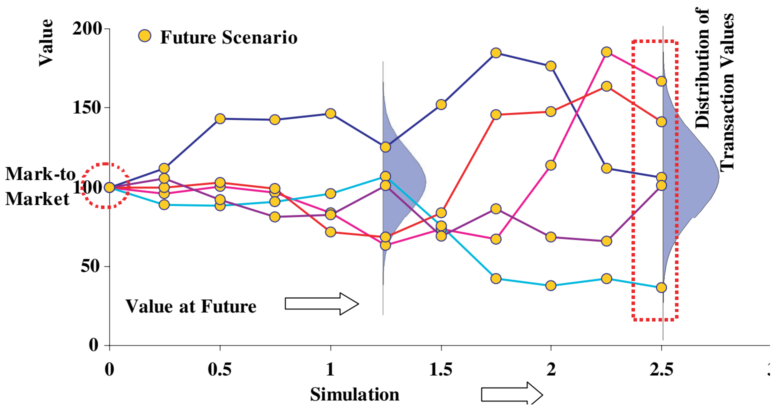

Figure 1: Mark-to-Market and Value-at-Future in Exposure Modeling

in this framework is that we simulate future scenarios only at *discrete* set of dates, while the value of the instrument may depend on the full continuous path prior to the simulation date (or a discrete set of dates different from the given set of simulation dates). Therefore, the valuation models used to calculate exposure could be very different from the front-office pricing models. For credit exposure calculations, pricing is not an end in itself. What is important is the distribution of instrument values (under the real measure) at different times in the future. The valuation models need to be optimized in order to perform sufficiently large number of calculations required to obtain such distribution. Term structure models[4] such as HW, HJM and BGM are not adequate for exposure calculation because these models require either Monte-Carlo or lattice-based modeling, which is computationally intensive. Furthermore, the standard valuation models used to price the instruments for mark-to-market are not applicable for calculating exposures on many path-dependent products whose value at the future time may depend on either some event at an earlier time (such as exercise of an option) or in some cases on the entire path leading to the future date (such as the case of knock-in and knock-out barriers). For such path-dependent instruments, we propose in this paper the notion of "conditional valuation", which is based on probabilistic conditional expectation techniques and develop value-at-future models for calculating the exposure of several path-dependent derivative instruments.

2. Scenario Generation

The first step in calculating credit exposure is to generate potential market scenarios (e.g., FX rates, equity prices, interest rates, etc.) at different times in the future. One obvious choice is to use the same model for instrument pricing and to generate the scenarios, but the evolution dynamics of this type of models are often constrained by arbitrage arguments. In contrast, the dynamics for risk measurement are usually built on a real measure based on historical data and not necessarily constrained to a risk neutral framework. For example, in a front-office pricing model, the interest rate scenarios are usually generated by construction[5] of zero rates or discount factors using the market prices of cash, euros and swap rates. However, such construction is often computationally expensive, as it requires the search algorithm for the business day count library. Furthermore, the forward rates implied from these scenarios can be nonsensical as a result of arbitrage-free constraints.

In this paper, we assume without loss of generality the simple lognormal model for underlying prices

$$S(t) = S_0 \exp\left[\left(\mu - \tfrac{1}{2}\sigma^2\right)t + \sigma W(t)\right] \tag{2.1}$$

where $W(t)$ is the Brownian motion, μ is the drift and σ is the volatility. However, distinction must be made between scenario generation and instrument valuation for these parameters. When the model is used for pricing or instrument valuation, we know that the volatility $\sigma = \sigma_{iv}$, i.e., the implied volatility for the option on underlying price, and the drift is set under the risk-neutral measure to $\mu = r - d$ for stock or indices with $r = \text{interest rate}$ and $d = \text{dividend yield}$. On the other hand, when the model is used for generating future scenarios for risk management, we normally use the drift $\mu = \mu_h$ and volatility $\sigma = \sigma_h$, which are usually estimated from the historical data as follows:

$$\sigma_h = \sqrt{\frac{1}{T}\sum_{t=1}^{T}\left(\ln\left[\frac{S(t)}{S(t-1)}\right] - \mu_h\right)^2},\ \mu_h = \frac{1}{T}\sum_{t=1}^{T}\ln\left[\frac{S(t)}{S(t-1)}\right] \tag{2.2}$$

and we then adjust the drift $\mu = \mu_h + \frac{1}{2}\sigma^2$ to compensate $-\frac{1}{2}\sigma^2$ term in the model (2.1).

There are two ways that we can generate the possible future values of the market factors. The first is to simulate directly from time $t = 0$ to the relevant simulation date t (i.e., Direct Jump to Simulation Date as shown in *Figure 2A*

$$X(t) = X(0)\exp[(\mu - \tfrac{1}{2}\sigma^2)t + z\sigma\sqrt{t}] \tag{2.3}$$

where z is a normal variant and $X(t)$ represents the shocked market factor at time t. The other method is to generate a "path" of the market factors through time (i.e., Path-Dependent Simulation as shown in *Figure* 2B

$$X(t_{i+1}) = X(t_i)\exp[(\mu - \tfrac{1}{2}\sigma^2)(t_{i+1} - t_i) + z\sigma\sqrt{t_{i+1} - t_i}] \tag{2.4}$$

where z is a standard normal variate and $X(t_{i+1})$ represents the shocked market factor at time t_{i+1} that connects from the particular scenario $X(t_i)$ at previous time t_i. Each simulation describes a possible trajectory from time $t = 0$ to the longest simulation time $t = T$.

Since continuous-time models are used for the market factor evolution (rather than simple discretization), the market factor distribution at

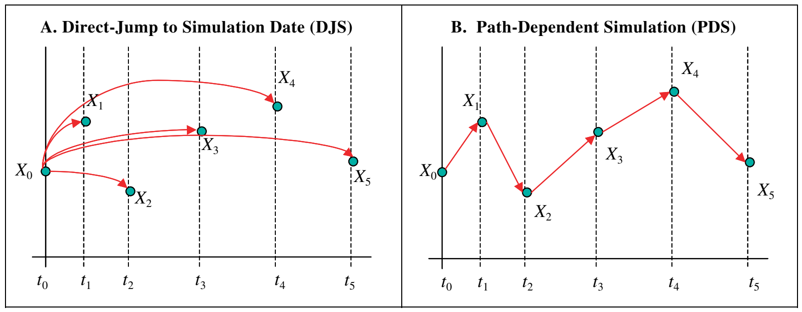

Figure 2: Two ways of generating market scenarios

a given simulation date using either PDS or DJS will be indistinguishable in the limit of large number of samples. Since we will be using conditional valuation methods in this paper, what matters is the distribution of the market factor scenarios at a simulation date rather than the path it took to get there.

3. The Brownian Bridge

The concept of conditional expectation can be illustrated by using the example of a Brownian bridge. The Brownian bridge is a set of Brownian paths $W(t)$ that start from one point (i.e., the origin) at time 0 and end at another point $W(T) = w$ pre-specified at a future time $T > 0$ as illustrated below:

The question of interest is to determine the distribution of $W(t)$ at any time $t \in (0, T)$ conditional on knowing the end point $W(T) = w$! Mathematically, the density of a Brownian bridge[6] can be explicitly derived such that

$$f(W(t) = x|W(T) = w) = \frac{1}{\sqrt{(2\pi)t(1-t/T)}} \exp\left[-\frac{(x - wt/T)^2}{2t(1-t/T)}\right] \quad (3.1)$$

where $W(t)$ is a standard Brownian motion[7]. Compared to standard Brownian motion, the Brownian Bridge possesses some unique properties. In particular, the conditional mean and variance of the Brownian Bridge are given by

$$\mathrm{E}[W(t)\,|W(T)\,] = (t/T)W(T) \quad (3.2)$$

$$\mathrm{Var}[W(t)\,|W(T)\,] = t(1 - t/T)$$

and for the geometric Brownian bridge,

$$\mathrm{E}[\exp\{W(t)\}|W(T)] = \exp\{\tfrac{1}{2}t(1 - t/T) + (t/T)W(T)\} \quad (3.3)$$

$$\mathrm{Var}[\exp\{W(t)\}|W(T)] = \exp\{2t(1 - t/T) + 2(t/T)W(T)\}$$

For exposure calculation, the conditional valuation is a probabilistic technique to adjust the valuation models for instruments whose value at the future date may depend on the scenario before that time. Such approach fits naturally in the value-at-future exposure framework, where one can separate instrument valuation completely from the market risk factor scenario generation. Furthermore, this approach provides a consistent treatment across different types of instruments, which then enable us to aggregate the exposures between the instruments of different types, such as swaps and swaptions. In the following sections, we will demonstrate the powerful techniques of conditional valuation approach in the exposure calculation for several well-known instrument types with the path-dependent features.

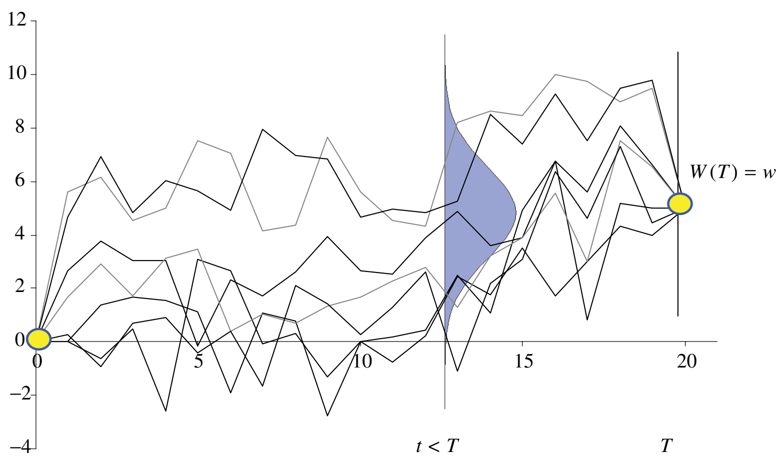

Figure 3: Graphical illustration of a Brownian bridge

4. Formulation of Conditional Valuation

We again emphasize that the need for conditional valuation stems from the fact that we can only simulate future scenarios at discrete time intervals because of limited computer resources. However, the value of a derivative product at any of these dates may depend on the full path over the continuum of dates prior to the simulation date. We provide in this section the formulation of the conditional valuation approach for calculating credit exposures that are consistent across all derivative products, path-dependent or not. For simplicity of exposition, we will assume in this paper that the direct-jump to simulation date (DJS) approach as explained in section 2 is used to generate market scenarios.

The exposure calculation is performed on a discrete set of the future times, which are called in this paper as the "simulation dates". As described in section 1, there are two steps in the calculation of future exposure of a derivative instrument. First, the future scenarios of market factors must be generated under the real measure (as opposed to the risk-neutral measure) on the simulation dates. Second, one needs to adjust the real measure to a risk-neutral measure when applying the valuation functions to calculate the value of derivative contract for each of the market scenarios on the simulation date. For this purpose, we first introduce the notations used to describe the evolution of market risk factor and exposure calculation:

- Discrete simulation dates: $\{t_k = t_1, t_2, \ldots, t_N\}$
- Market risk factor scenarios: $\{X(t_k) = X(t_1), X(t_2), \ldots, X(t_N)\}$
- The future values of the transaction: $\{V(t_k) = V(t_1), V(t_2), \ldots, V(t_N)\}$

Since the credit exposure of a derivative contract at a future time depends on the expected value of the contract given the scenario of underlying market risk factors at that time, we need a valuation methodology that can calculate the future value of derivative contract which may be contingent on the scenarios of underlying market risk factors between today and the future time. The scenarios can be generated from the Monte-Carlo simulation of a risk factor evolution model in two different ways: path-dependent scenarios and direct-jump scenarios, as described in section 2.

There are many situations where the future value of a given transaction is not uniquely determined by the state of underlying risk factors at the simulation date. For example, we consider a swap-settled swaption (or a Bermudan in a general case) where the future value of such transaction can be ambiguous on the simulation date past the expiry date of option, because we could either have a swap as the result of option exercised or nothing if the swaption expires worthless. Other examples include barrier (i.e., knock-in or knock-out) and average options, where the payoff is truly path-dependent in the sense that the future values of such options at the simulation date depend on the entire history of underlying market factor. For path-dependent instruments, their values at the future simulation date may thus depend on either the event occurring at a time before the simulation date or the entire scenario path leading to the simulation date. Hence, the valuation at the future date can be formulated as the conditional expectation

$$V(t_k, x) = \mathrm{E}[f(t_k, \{X(t)\}_{0\le t\le t_k}) \,|X(t_k) = x] \tag{4.1}$$

where the conditioning is on the state of market risk factor $X(t_k) = x$, for a particular scenario at t_k and $\{X(t) : 0 \le t < t_k\}$ denotes the entire path of risk factor evolution. When an instrument is not path-dependent, such as in the case of swap, forward and cash-settled option, the conditional expectation in (4.1) above simply degenerates to

$$V(t_k, x) = f(t_k, X(t_k) = x) \tag{4.2}$$

which is just a simple MtM valuation at the simulation date t_k.

In this paper, we refer to this type of valuation as the "conditional value-at-future" or simply value-at-future (VaF) as described in section 1. However, in general, VaF is not the MtM valuation at the future simulation date. To illustrate the difference, we consider two special cases in the formulation (4.1) where the valuation function is separable in the following sense:

$$f(t_k, \{X(t)\}_{0\le t\le t_k}) = g(t_k, X(t_k)) \cdot h(\{X(t)\}_{0\le t\le t_k}) \tag{4.3}$$

$$f(t_k, \{X(t)\}_{0\le t\le t_k}) = g(t_k, X(t_k)) + h(\{X(t)\}_{0\le t\le t_k}) \tag{4.4}$$

In each of two cases, we can respectively rewrite the conditional expectation explicitly

$$V(t_k, x) = g(t_k, x) \cdot \mathrm{E}\{h(\{X(t)\}_{0\le t\le t_k}) \,|X(t_k) = x\} \tag{4.5}$$

$$V(t_k, x) = g(t_k, x) + \mathrm{E}\{h(\{X(t)\}_{0\le t\le t_k}) \,|X(t_k) = x\} \tag{4.6}$$

where $g(t_k, x)$ is the mark-to-market valuation of such transaction at the simulation date. As shown later in this paper, the barrier option is an example of the case (4.3) where

$$h(\{X(t)\}_{0\le t\le t_k}) = I_{\{X(t)<H:\ 0\le t\le t_k\}} \tag{4.7}$$

is the indicator function of breaching the up barrier. The average option is an example of the case (4.4) where

$$h(\{X(t)\}_{0\le t\le t_k}) = \sum_{i=0}^{k} \frac{X(t_i)}{(t_k - t_0)} \tag{4.8}$$

is the average of the scenario history leading to t_k.

Finally, the formulation of conditional valuation in (4.1) provides the consistency for transactions with and without path-dependence, and thus makes it possible for netting and aggregation across multiple risk factors. Such approach is relatively easy to implement, as it is feasible in many cases to explicitly compute the conditional expectation for many instruments such as the barrier option, average option, swaption and variance swap.

5. Barrier Option

The barrier options are typical examples of path-dependent options where their payout at maturity is determined by the entire history of the underlying asset prices. They are often embedded in the interest rate products such as the knockout swap, knockout cap and floor, where the swaps, caps or floors will cease to exist if the forward rate rises or falls below a pre-specified level (i.e., the barrier).

The pricing of barrier options is well known, and their (risk-neutral) values are given by

$$MtM_{barrier}(t) = \begin{cases} \mathrm{E}_t[\max\{0, S(T) - K\} \cdot I_{\{S_{Max}(t,T)<H\}}], & \text{for up-out Call} \\ \mathrm{E}_t[\max\{0, K - S(T)\} \cdot I_{\{S_{Min}(t,T)>L\}}], & \text{for down-out Put} \end{cases} \tag{5.1}$$

where $S_{Max}(t, T) = \max\{S(\tau), t < \tau \le T\}$ and $S_{Min}(t, T) = \min\{S(\tau),\ t < \tau \le T\}$, E_t denotes the expectation under the risk-neutral measure, H = the up barrier and L = the down barrier. If the market risk factor $S(t)$ is assumed to follow a (risk-neutral) lognormal process,

$$S(t) = S_0 \exp[-\tfrac{1}{2}\sigma^2 t + \sigma \bar{W}(t)] \tag{5.2}$$

then the analytic solutions such as Black-Scholes type formula to (5.1) can be found in Haug and Espen (1997) and Hull and John (2003). However, our interest here is to calculate the exposures of these instruments at some future simulation date.

For exposure calculation, we first specify the evolution of risk factor in the actual measure

$$S(t) = S_0 \exp\left[\left(\mu - \tfrac{1}{2}\sigma^2\right) t + \sigma W(t)\right] \tag{5.3}$$

which contains a drift and the volatility that are different from the risk-neutral process in (5.2). The drift, which represents the risk premium for the future uncertainty, can be calibrated to the historical time series.

For a fixed scenario $S(t_k) = x$, we illustrate four types of paths in the figure shown below:

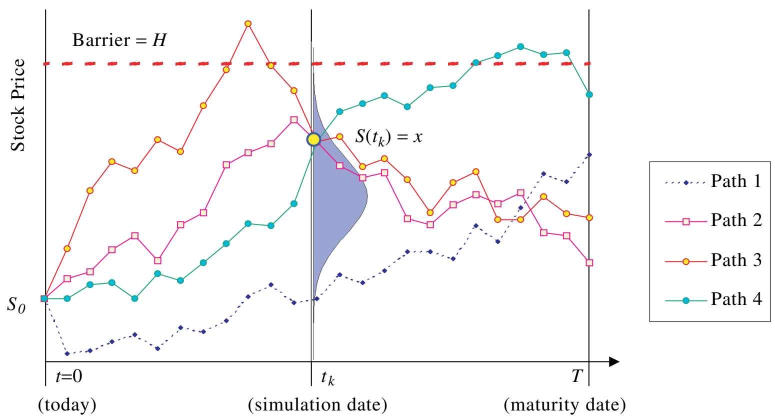

Figure 4: Different scenario paths for crossing the barrier

- Path 1 did not pass through the fixed scenario at the simulation date
- Path 2 never breached the barrier before the maturity,
- Path 3 breached the barrier before the simulation date,
- Path 4 breached the barrier before the maturity but after the simulation date.

Thus, the third path will affect the calculation of exposure at the simulation date since the crossing of barrier before the simulation date will determine the existence or extinction of underlying option.

Since the digital options are simple cases of barrier options, we describe only the conditional valuation of barrier options, particularly the up-out and down-out call options. The value-at-future calculation of such barrier options are given by the conditional expectation

$$\begin{aligned}\mathrm{VaF}_{uo}(t_k; x) &= \mathrm{E}[\max\{0, S(T) - K\} \cdot I_{\{S_{Max}(t_k,T)<H\}} \cdot I_{S_{Max}(0,t_k)<H} \,|S(t_k) = x] \\ &= \mathrm{E}[\max\{0, S(T) - K\} \cdot I_{\{S_{Max}(t_k,T)<H\}}] \times \mathrm{E}[I_{\{S_{Max}(0,t_k)<H\}} |S(t_k) = x] \\ &= MtM_{uo}(t_k; x) \times \mathrm{Prob}[S_{Max}(0, t_k) < H \,|S(t_k) = x] \end{aligned} \tag{5.4}$$

for the up-out barrier options, and similarly we have

$$\begin{aligned}\mathrm{VaF}_{do}(t_k; x) &= \mathrm{E}_t[\max\{0, S(T) - K\} \cdot I_{\{S_{Min}(t_k,T)>L\}} \cdot I_{\{S_{\min}(0,t_k)>L\}} \,|S(t_k) = x] \\ &= \mathrm{E}[\max\{0, S(T) - K\} \cdot I_{\{S_{Min}(t_k,T)>L\}}] \times \mathrm{E}[I_{\{S_{Min}(0,t_k)>L\}} |S(t_k) = x] \\ &= MtM_{do}(t_k; x) \times \mathrm{Prob}[S_{Min}(0, t_k) > L \,|S(t_k) = x] \end{aligned} \tag{5.5}$$

for the down-out barrier options. Thus, the VaF calculation is simply the mark-to-market value at the fixed scenario $S(t_k) = x$ multiplied by the conditional probability of crossing the barrier, which can be computed using the so-called "reflection principle" of Brownian path as described in Karatzas and Shreve (1991) such that

$$\mathrm{Prob}[S_{Max}(0, t_k) < H \,|S(t_k) = x] = 1 - \exp[2h_k(x_k - h_k)], \tag{5.6}$$

and

$$\mathrm{Prob}[S_{Min}(0, t_k) > L \,|S(t_k) = x] = 1 - \exp[2l_k(x_k - l_k)], \tag{5.7}$$

where $x_k = \log[S(t_k)/S_0]/(\sigma\sqrt{t_k})$, $h_k = \log(H/S_0)/(\sigma\sqrt{t_k})$ and $l_k = \log(L/S_0)/(\sigma\sqrt{t_k})$, note the above probabilities are well defined since $l_k \le x_k \le h_k$.

Finally, we consider an example of an up-out barrier option with one-year maturity and the barrier level at 10% above the ATM strike. Using (5.4), we compute the exposure profiles at 95%, 50% and 5% confidence levels respectively on different simulation dates over a one year period as shown in the figure below. Note that the exposure profiles of the barrier option are quite different compared to that of a standard option. The exposure profile of an up-out barrier option exhibits a sharp convexity in contrast to the concave profile of a vanilla option. The peaky shape of exposure profile on an up-out barrier option is attributed to the negative convexity that the option becomes more likely to be knocked out as the underlying price reaches close to the barrier level.

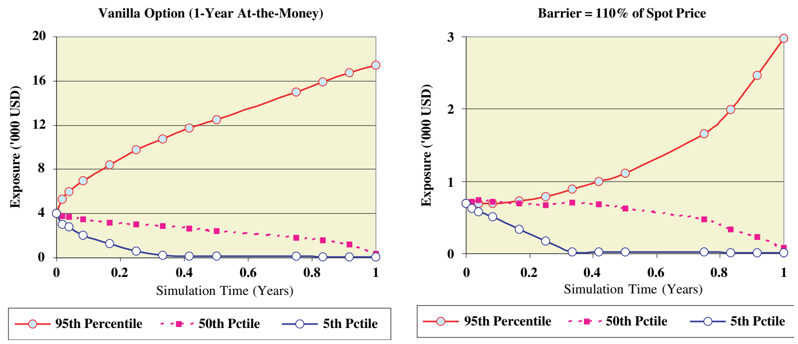

Figure 5: Exposure Profile for Up-Out Barrier at 95-50-5% Confidence

6. Asian Option

An Asian option is an option on the average of underlying prices or interest rates taken at certain frequency (such daily or weekly) from the start date to the maturity date. There are generally two types of averaging methods: arithmetic average and geometric average. The arithmetic average is defined as

$$A_n = \sum_{i=1}^{n} S(t_i)/n \tag{6.1}$$

while the geometric average is defined as

$$G_n = \prod_{i=1}^{n} S(t_i)^{1/n} \tag{6.2}$$

where $S(t_i)$ is the price of underlying at reset time t_i. The price of underlying asset is assumed to follow a lognormal process

$$S(t) = S_0 \cdot \exp[(\mu_\lambda - \tfrac{1}{2}\sigma^2)t + \sigma W(t)] \tag{6.3}$$

where σ is the volatility of underlying stock, $\mu_\lambda = r - d + \lambda\sigma$ is the risk-adjusted drift for a specified market price of risk λ. The geometric average, as a product of underlying prices, is lognormal since the underlying price is lognormal at each reset time. However, the arithmetic average as the sum of lognormal prices will not be lognormal in general. Nonetheless, to handle the Asian feature, we approximate the arithmetic average with a lognormal distribution with mean and variance chosen to match the actual mean and variance. When average reset frequency is high such as daily, the lognormal volatility of arithmetic average can be approximated by so-called the "$\sqrt{3}$-rule". This approximation rule is derived in the following:

$$\begin{aligned}\mathrm{Var}[A_n] &= \frac{1}{n^2}\mathrm{Var}[nS(t_0) + (n-1)S(t_1) + \cdots + S(t_n)] \\ &= (T - t_0)\sigma^2 \frac{(n-1)^2 + \cdots + 2^2 + 1^2}{n^3} \\ &= (T - t_0)\sigma^2 \frac{(n-1)(2n-1)}{6n^2}\end{aligned} \tag{6.4}$$

Annualizing by $(T - t_0)$ and taking the limit $n \to \infty$, we obtain $\mathrm{Var}[A_n]/(T - t_0) \approx \sigma^2/3$.

In general, the mean and variance of arithmetic average can be calculated in the following

$$\mathrm{E}[A_n] = \frac{1}{n}\sum_{i=1}^{n} F_0(t_i),\ \mathrm{Var}[A_n] = \mathrm{E}[A_n^2] - \mathrm{E}[A_n]^2 \tag{6.5}$$

where $F_0(t) = S_0 \exp\{(r-d)t\}$ is the forward price at today for delivery at the future date t. Furthermore, we define the variance of return on arithmetic average

$$\sigma_A^2 = \frac{1}{T}\left(1 + \frac{\mathrm{Var}(A_n)}{(\mathrm{E}[A_n])^2}\right) \tag{6.6}$$

The option on average price can generally be classified into two different types: average price option (APO) and average strike option (ASO). The payoff on average price option is defined as

$$\max\{0, \phi(A_n - K)\}$$

while the payoff on average-strike option

$$\max\{0, \phi(S(T) - A_n)\}$$

where $\phi = +1$ for a Call and $\phi = -1$ for a Put. That is to say, the APO is an option strike on a fixed price and the ASO is an option strike on the average price.

For valuation of a European option on arithmetic average, we usually approximate by a Black-Scholes function assuming a lognormal average price:

$$MtM(t_0) = \mathrm{E}[\max\{0, \phi(A_n - K)\}] = \phi M \cdot \Phi(\phi d_1) - \phi K \cdot \Phi(\phi d_2) \tag{6.7}$$

where $M = \mathrm{E}[A_n]$, σ_A as calculated in (6.6), $\Phi(\cdot)$ is cumulative normal distribution and

$$d_1 = \frac{\ln(M/K) + 0.5\sigma_A^2\sqrt{T}}{\sigma_A\sqrt{T}}, \quad d_2 = d_1 - \sigma_A\sqrt{T}$$

At any future simulation date $t_k > t_0$, we denote

$$A(t_0, t_k) = \begin{cases} \frac{1}{n_k}\sum_{i=1}^{k} S(t_i) \\ \frac{1}{t_k - t_0}\int_{t_0}^{t_k} S(t)dt \end{cases} \tag{6.8}$$

as the discrete or continuous average between t_0 and t_k respectively, and similarly for $A(t_k, T)$ as the average between t_k and T. Then the value-at-future of average option is given by the conditional expectation

$$\begin{aligned}\mathrm{VaF}(t_k; S(t_k)) &= \mathrm{E}[\max\{0, \phi(A(t_0, T) - K)\}|S(t_k)] \\ &= \mathrm{E}\left[\max\left\{0, \phi\left(\frac{n_k}{n}A(t_0, t_k) + \frac{n-n_k}{n}A(t_k, T) - K\right)\right\}\middle|W(t_k)\right]\end{aligned} \tag{6.9}$$

where the expectation is taken conditional on the price of underlying stock $S(t_k)$ at simulation time. Since $A(t_k, T)$ is independent of $W(t_k)$, the conditional expectation in (6.9) can be rewritten as

$$\begin{aligned}\mathrm{VaF}(t_k; S(t_k)) = \frac{n-n_k}{n}\mathrm{E}\Bigg[\mathrm{E}\Bigg(\max\Bigg\{0, \phi(A(t_k, T) \\ -\frac{n}{n-n_k}\left[K - \frac{n_k}{n}A(t_0, t_k)\right]\Bigg)\Bigg\}\Bigg)\Bigg|W(t_k)\Bigg]\end{aligned} \tag{6.10}$$

There is no analytic solution for the conditional expectation, in general, for arithmetic average option. However, we can find a good semi-analytic approximation by assuming a lognormal distribution for the averages:

$$A(t_0, t_k; z) = \exp\{M(t_0, t_k) + \sqrt{V(t_0, t_k)} \cdot z\}$$

$$A(t_k, T; x) = \exp\{M(t_k, T) + \sqrt{V(t_k, T)} \cdot x\}$$

where M is the mean, V is the variance of the average, x and $z \sim$ N(0,1) with

$$\langle X, Z\rangle = \rho(t_k) = \mathrm{Corr}[\ln A(0, t_k), \ln A(t_k, T))] \tag{6.11}$$

The mean and variance can be computed once their 1st and 2nd moments are known

$$A = 2\ln(M1) - \tfrac{1}{2}\ln(M2),\ V = \ln(M2) - 2\ln(M1) \tag{6.12}$$

For two moments over time interval $[t_k, T]$, we can compute in straightforward fashion

$$M1(t_k, T; x) = \frac{S(t_k)}{n}\sum_{j=1}^{n-n_k}\exp[\mu \cdot j \cdot \Delta t] \tag{6.13}$$

$$\begin{aligned}M2(t_k, T; x) = \left[\frac{S(t_k)}{n}\right]^2 \cdot \sum_{j=1}^{n-n_k}\Bigg[\exp\left\{\frac{(2\mu + \sigma_{iv}^2)j}{n}\right\} \\ +2 \cdot \sum_{i=1}^{j-1}\exp\left\{\frac{\mu(j+i)}{n} + \frac{\sigma_{iv}^2(j+3i)}{2n}\right\}\Bigg]\end{aligned} \tag{6.14}$$

However, the calculation of the two moments over the time interval $[t_0, t_k]$ is quite complex, as the repeated use of the variance formula (3.3) of the Brownian Bridge is required

$$\begin{aligned}M1(t_0, t_k; z) &= \frac{1}{n+1}\mathrm{E}\left[\sum_{j=1}^{n_k} S(t_j)|S(t_k)\right] = \frac{1}{n+1}\sum_{j=1}^{n_k}\mathrm{E}\left[S(t_j)|S(t_k)\right] \\ &= \frac{S_0}{n+1}\left[1 + \sqrt{2\pi} \cdot \exp\left\{\frac{1}{2}\left(\frac{\mu\sqrt{t_k}}{\sigma_h} + z\right)^2\right\} \right. \\ &\quad \left. \cdot \sum_{j=1}^{n_k}\psi\left(\frac{\mu\sqrt{t_k}}{\sigma_h} + z - \frac{\sigma_h j\Delta t}{\sqrt{t_k}}\right)\right]\end{aligned} \tag{6.15}$$

and

$$
\begin{aligned}
M2(t_0, t_k; z) &= \frac{1}{(n+1)^2} \mathrm{E}\left[\left(\sum_{j=1}^{n_k} S(t_j)\right)^2 |S(t_k)\right] \\
&= \frac{1}{(n+1)^2}\left(\sum_{j=1}^{n_k} \mathrm{E}[S(t_j)^2|S(t_k)] + 2\sum_{i=1}^{j-1} \mathrm{E}[S(t_j)S(t_i)|S(t_k)]\right) \\
&= \frac{S_0^2}{(n+1)^2}\left[\begin{array}{l} 1+\sqrt{2\pi}\cdot\exp\{((\mu+0.5\sigma_h^2)t_k + z\sigma_h\sqrt{t_k})^2/(2\sigma_h^2 t_k)\} \\ \cdot \sum_{j=1}^{n_k}\left[\begin{array}{l} \psi\left(\dfrac{(\mu+0.5\sigma_h^2)t_k + z\sigma_h\sqrt{t_k} - 2\sigma_h^2 j\Delta t}{\sigma_h\sqrt{t_k}}\right) \\ +2\cdot\sum_{i=1}^{j-1}\exp\{i\Delta t\sigma_h^2\} \\ \cdot\psi\left(\dfrac{\mu t_k + z\sigma_h\sqrt{t_k} - 2\sigma_h^2(j+i)\Delta t}{\sigma_h\sqrt{t_k}}\right) \end{array}\right] \end{array}\right]
\end{aligned}
\tag{6.16}
$$

Finally, we apply the Black-Scholes option formula to the unconditional expectation in (6.10)

$$
\begin{aligned}
\mathrm{VaF}(t_k; S(t_k)) = \frac{n-n_k}{n}\int_0^{K_k} \psi(z)dz\cdot \\
\mathrm{BS}[A(t_k, T; z), K_k(z), T-t_k, \sqrt{(1-\rho^2(t_k))V(t_k, T; x)}, \phi] \\
+\phi\int_{K_k}^{\infty}\left(\frac{n_k}{n}A(t_0, t_k; z) + \frac{n-n_k}{n}A(t_k, T; z) - K\right)\psi(z)dz
\end{aligned}
\tag{6.17}
$$

and

$$
\begin{aligned}
&\int_{K_k}^{\infty}\left(\frac{n_k}{n}A(t_0, t_k; z) + \frac{n-n_k}{n}A(t_k, T; x) - K\right)\psi(z)dz \\
&= \left[\frac{n_k}{n}\Phi\left(-K_k + \sqrt{V(t_0, t_k; z)}\right) + \frac{n-n_k}{n}\Phi\left(-K_k + \rho(t_k)\sqrt{V(t_0, t_k; z)}\right)\right] \\
&\quad - K\Phi(-K_k)
\end{aligned}
\tag{6.18}
$$

where ψ is the standard normal density function, Φ is the cumulative distribution function and $\mathrm{BS}[X, K, T, \sigma, \phi]$ is the standard Black-Scholes option pricing function,

$$
\begin{aligned}
K_k &= [\ln(K\cdot N/n_k) - M(t_0, t_k)]/\sqrt{V(t_0, t_k)} \\
K(t_k, Z) &= \frac{N}{N-n_k}\left[K - \frac{n_k}{N}\exp\left\{M(t_0, t_k) + \sqrt{V(t_0, t_k)}\cdot Z\right\}\right] \\
A(t_k, T; Z) &= \exp\left\{M(t_k, T) + \tfrac{1}{2}V(t_k, T)(1-\rho^2(t_k)) + \sqrt{V(t_k, T)}\cdot Z\cdot\rho(t_k)\right\}
\end{aligned}
$$

The above valuation needs to perform an integration of the Black-Scholes function with a standard normal density function, which can be calculated using a simple numerical integration. As an example, we compute the exposure profiles on a 1.25-year average price option with weekly averaging frequency and a fixed strike set equal to the spot price. Compared with a standard ATM option, the average option has a lower peak exposure and the exposure profile of such option exhibits a humped shape in the rising price scenarios as shown in Figure below.

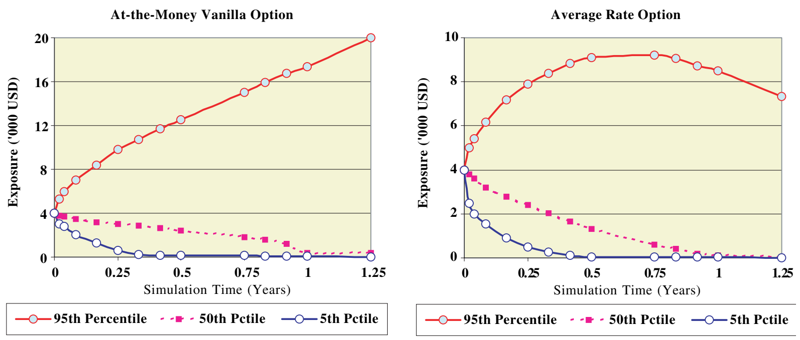

Figure 6: Exposure Profile for Asian Option vs. Standard Option

7. Swap-Settled Swaption

An interest rate swaption is an option to enter a swap. Depending on the swap being a payer or receiver, such option is usually called right-to-pay (RTP) or right-to-receive (RTR) swaption. In contrast to regular swaptions where the option will be cash-settled at the expiration date, a swap-settled swaption will settle into its underlying swap if the option is exercised at the expiration of the option.

For credit exposure, this difference is very crucial as the future exposure on a cash-settled swaption stops right before the expiry while the future exposure on a swap-settled swaption can potentially continue well beyond the expiry of the option into the remaining life of underlying swap.

The payout at option expiration date is defined as

$$
MtM_{swptn}(T_e) = \max\{0, \phi\cdot Swap(T_e)\} \tag{7.1}
$$

where

$$
Swap(T_e) = \left(\sum_{i=1}^{N} b_i d_i\right)\phi[F(T_e, T_i) - K]. \tag{7.2}
$$

where K is the fixed rate, $F(T_e, T_i)$ is the forward rate reset at ith swap period, $\phi = 1$ for a call option or RTP swaption and $\phi = 1$ for a put option or RTR swaption. To price a swaption (both cash-settled and swap-settled), Black's formula is usually applied to the forward swap rate of the underlying swap in practice, i.e.,

$$
\begin{aligned}
MtM_{swptn}(t) &= \left(\sum_{i=1}^{N} b_i d_i\right)\cdot \mathrm{E}_S\{\max[0, \phi S(T_e) - \phi K]\} \\
&= \left(\sum_{i=1}^{N} b_i d_i\right)\cdot \mathrm{BS}(S_0, K, T_e - t, \sigma_S, \phi)
\end{aligned}
\tag{7.3}
$$

where N is the number of swap periods, b_i is the day-count fraction and d_i is the discount factor. Here, σ_S is the implied volatility and E_S denotes the expectation under the forward swap measure as explained in Hull [5], such that $\mathrm{E}_S[S(T_e)] = S_0$ at expiry time T_e and the swap rate $S(t)$ follows a lognormal process under the forward measure,

$$
S(t) = S_0\exp[-\tfrac{1}{2}\sigma_S^2 t + \sigma_S\bar{W}(t)]. \tag{7.4}
$$

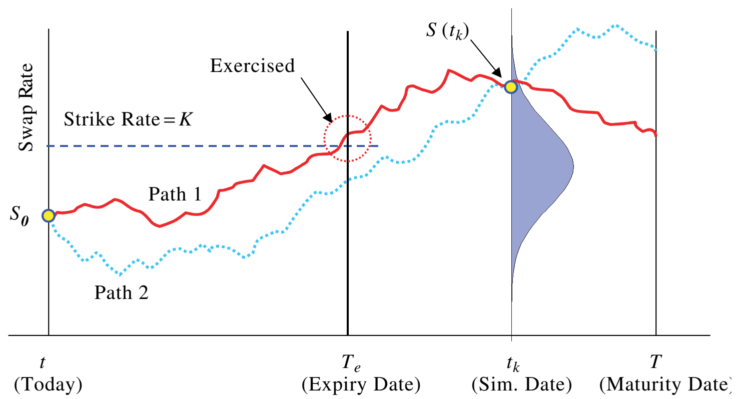

Figure 7: Option Expiries in a Swap-Settled Swaption

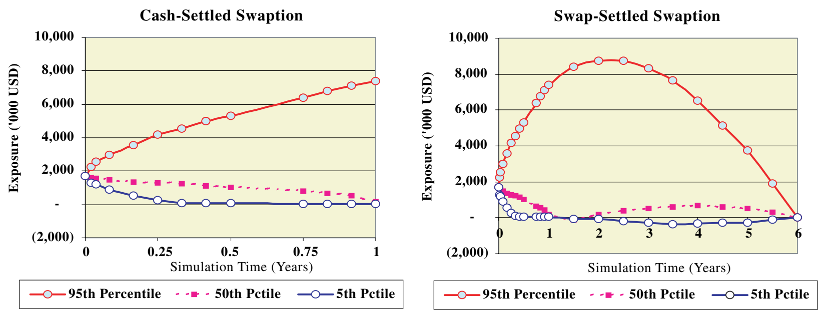

Figure 8: Exposure Profiles of Cash-Settled and Swap-Settled Swaptions

For exposure calculation, we choose the swap rate as the risk factor and model its evolution under the actual measure as

$$S(t) = S_0 \exp\left[\left(\mu - \tfrac{1}{2}\sigma^2\right)t + \sigma W(t)\right] \tag{7.5}$$

for some drift μ and volatility σ that are calibrated to the historical time series. For any future time before the expiry T_e, the exposure is simply given by the valuation formula (7.3).

For swap-settled swaptions, note that when the simulation date t_k is past the expiry T_e, we face an ambivalent situation whether or not to calculate the exposure on an underlying swap since we are not sure if the swaption was exercised earlier. To illustrate this, consider two paths leading to the fixed scenario as in *Figure 7*. Path 1 (solid line) implies an option exercise into the underlying swap since the swap rate at expiry T_e is above the strike rate. However, Path 2 (dotted line) implies that the option expires worthless. Therefore, the calculation of the exposure at time $t_k > T_e$ should include the probability of option exercise.

Thus, the value-at-future of such swaption is given by the conditional expectation

$$\begin{aligned} \mathrm{VaF}_{swptn}(t_k) &= \left(\sum_{i=1}^{N} b_i d_i\right) \cdot \mathrm{E}_S\{Max[0, \phi S(T_e) - \phi K]\,|S(t_k)\} \\ &= \left(\sum_{i=1}^{N} b_i d_i\right) \cdot \phi[S(t_k) - K] \times \mathrm{Prob}\{\phi S(T_e) > \phi K|S(t_k)\} \end{aligned} \tag{7.6}$$

for the future time $t_k > T_e$. The conditional probability can be computed by applying the Brownian Bridge to the swap rate evolution up to the fixed scenario $S(t_k)$. The risk factor evolution after the simulation date t_k needs to be adjusted back to the risk-neutral process (7.4) for valuation. For the fixed swap rate scenario, the swap rate at expiry T_e can be expressed in term of a Brownian bridge such that

$$S(t_k) = S(T_e) \cdot \exp[(\mu - \tfrac{1}{2}\sigma^2)(t_k - T_e) + z \cdot \sigma\sqrt{t_k - T_e}]$$

under the actual measure. This enables us to compute the conditional probability of option exercise at the expiration time

$$\mathrm{Prob}\{\phi S(T_e) > \phi K|S(t_k)\} = 1 - \Phi[\phi z^*(T_e, t_k)] \tag{7.7}$$

where $z^*(T_e, t_k) = \dfrac{\{\ln[S_0/K] - (T_e/t_k)\ln[S(t_k)/S_0]\}}{\sigma\sqrt{t_k(1 - T_e/t_k)}}$, using the property of Brownian bridge (3.3).

Hence, the value-at-future after the expiration date is the product of the MtM value of remaining swap and the conditional probability of option exercise at expiry.

As an example, we consider both the cash-settled and swap-settled RTP swaption striked at the money with 1-year option to enter into a 5-year payer swap on USD 100m notional. The figures above compare the exposure profiles between these two swaptions. The cash-settled swaption reaches the peak exposure as expected at the option expiry date and the exposure does not extend beyond the expiration date, while the exposure of a swap-settled swaption goes beyond the option expiry and extends to the final maturity date of underlying swap and it reaches the peak exposure after the expiry date approximately at 3rd of underlying swap life.

8. Conclusion

The accurate calculation of exposure is an essential component in managing credit risk with the trading counterparties. In this paper, we have presented a technique using conditional expectation valuation to improve the accuracy of simulated exposures for path-dependent products. Since a typical counterparty portfolio generally consists of many types of instruments as well as a variety of market factors affecting the values of these instruments, the simulation approach as described in Duffie and Canabarro (2004) is usually employed to calculate exposure across the future time horizons, usually up to the longest maturity in the portfolio.

In the simulation approach, each transaction is revalued at future times using simulated future scenarios of market factors. However, simulation and revaluation consumes so much computer resources that certain simplifications are necessary in order for a bank to have its daily exposure report delivered to its users in a reasonable amount of time. One simplification that is mentioned in this paper is to have discrete sets of simulation dates, usually with increasing time intervals for the longer maturities. However, this simplification creates some difficulties in valuing a path-dependent instrument at a future simulation date since we do not have the continuum of the evolution of market factors up to that date. Examples are given in this paper of path-dependent instruments that depend on the history of a market factor leading up to its value at a given simulation date. Front-office pricing models are clearly inadequate to calculate the value-at-future of a path-dependent instrument since these models assume that no previous contingent event has taken place prior to the valuation date. In this paper we have presented a methodology to account for the possibilities of particular prior events that may affect the exposure, conditional on the simulated value of the relevant market factor at a given simulation date. Furthermore, using the properties of the Brownian Bridge, we have derived analytic expressions to calculate the exposure or value-at-future on a number of path-dependent instruments such as barrier options, average options, and swap-settled swaptions.

FOOTNOTES & REFERENCES

1. The Expected Exposure (EE), as defined in Basel (2005), is the mean of the distribution of exposures at any particular future date before the longest-maturity transaction in the netting set matures. The Expected Positive Exposure (EPE) is the time-weighted average of the individual expected exposures estimated for given forecast horizons.
2. The notion and concept of VaF is similar to Mark-to-Future (MtF), the terminology originally introduced by Ron Dembo, et al, in their 2000 paper (2000) for Algorithmics risk management framework.
3. Duffie and Canabarro (2004) provided a description of Monte-Carlo simulation approach on modeling the derivatives exposure.
4. Rebonato (2003) provided an excellent overview of interest rate term structure models.
5. The construction of zero curves is commonly referred as curve construction, which is a necessary step in pricing most interest rate instruments.
6. Brownian Bridge and its density can be easily extended to the case of multi-dimensional Brownian process.
7. We will use the notation W(t) to denote Brownian motion under the real (historical) measure and $\overline{W}(t)$ for risk-neutral measure.

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